ANALOG DIFFERENTIATION FOR CONTROL SYSTEMS

A THESIS

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ABSTRACT

This paper examines the standard methods for analog differentiation. It concludes that the compromised derivative circuit with paralled capacitance and resistance in the feedback loop and added resistance in the input leg is the most satisfactory normal circuit provided that the poles generated by the components are suitably located. It shows furthermore that multipole derivative circuits can be devised from mathematical approximations of the type

$$\frac{A^m y^{(m)}}{m!} = \frac{1}{p!} \quad (A_0 y_0 + A_1 y_1 + \ldots A_p y_p) + \mathcal{E}$$

Such circuits have greater descrimination against frequencies above a specific level and give more accurate derivate for frequencies which are low relative to this cut-off point.

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I. INTRODUCTION

The importance of quantitative description has been well stated by Lord Kelvin.

> "I often say that when you can measure what your are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind; it may be the beginning of knowledge, but you have scarcely, in your thoughts, advanced to the stage of science, whatever the matter may be."

Today in experimental engineering, in system and process control, we are concerned with the measurement and use of electrical signals derived from a wide range of varying physical states. Transducers exist which can convert the variation of measurable quantitites into such signals. However, at the present time, with the exception of velocity type pick ups, there are no transducers which give outputs proportional to the time rate of change of the transduced property. Likewise in many systems used for process control a derivative feed back loop is preferable to an integral feed forward. The factors of primary importance in connection with rate mode are that by opposing all change, rate mode has a great stabilizing effect on control.[1]*

The need for a clear understanding of electrical analog derivative circuits is apparent. On the assumption that the signal input is clean, standard text books on analog methodology give circuits which will produce derivative outputs. However, the world of reality, as we know it, is always noisy, and consequently this theoretical ideal input is never achieved. Thus, the derivative circuit responding to the noisy input differentiates both the noise and the pertinent signal. Since in many natural phenomena and control processes the variable changes are restricted to relatively low frequency and noise is primarily generated or picked up at high frequency the signal to noise ratio is worse after differentiation. This is due to the fact that a derivative device is by

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*Superscript numerals refer to the Bibliography at the end of this report.

definition a rate of change responding device. Standard authorities make it clear that the very nature of the response and of the electronic circuitry makes this unsatisfactory situation one which the process control and instrumentation engineers must learn to live with, to avoid or to compromise. Engineering is of course the art of compromise — but for such to be effective its implication must be understood. The purpose of this paper is to attempt to demonstrate how analog derivative circuits can be adapted to specific performance requirements.

II. STANDARD DERIVATIVE CIRCUITS

The circuits given in Figs 1a-c are those generally quoted by authoritative texts as the most suitable for analog differentiation.[2,3,4] The circuit shown in Fig. 1a is a true differentiator. This is readily see from the system analysis given below.

$$e_0(s) = -SRCe_{in}(s) \tag{1}$$

which may be written in the time domain

$$e_0(t) = -RC \cdot \frac{d}{dt} e_{in}(t) \tag{2}$$

This latter equation is of course a special case of the more general Laplace transform theorem.

$$\mathcal{L}\left(\frac{d^n f}{dt^n}\right) = s^n F(s) - \sum_{k=1}^{k=n} \frac{d^{k-1} f(0)}{dt^{k-1}} s^{n-k} \tag{3}$$

The validity of Eq. (2) is not dependent upon the form of the input signal, it merely requires that $n = 1$. Thus, the circuit will not discriminate against any frequency, but will give the derivative of the input which it receives. Let us consider then its behavior when the input is defined by

$$e_{in} = A \sin \omega t \tag{4}$$

$$e_{out} = -\omega RCA \cos \omega t \tag{5}$$

We see straightaway that the sinusoidal signal is phase shifted $\pi/2$ and increased in amplitude by the factor ωRC. Consider

$$e_{in} = f(t) + A \sin \omega t \tag{6}$$

and

$$e_{out} = f'(t) + \omega A \cos \omega t \tag{7}$$

It is clear that unless $f(t)$ is some simple function of ωt the relative amplitudes of the two components are not the same in the initial

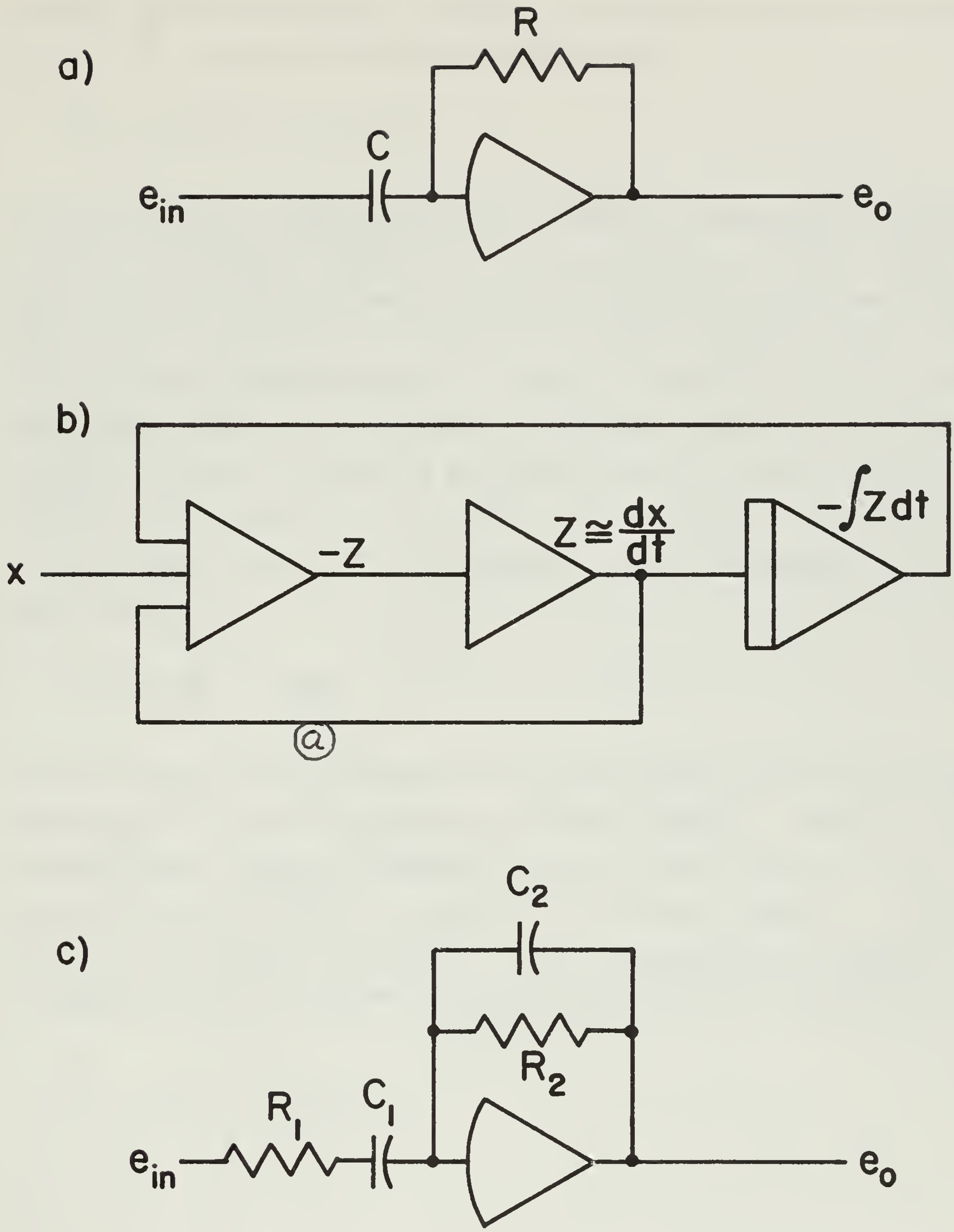

FIG. 1. DERIVATIVE CIRCUITS

function and the derivative. However, in the particular case for which $f(t) = e^{-\omega t}$ we meet the special condition that

$$-e_{out} = \omega f(t) + \omega A \cos \omega t \tag{8}$$

In analog control loops the pertinent signals are generally restricted to the low-frequency range but this essential data is almost always mixed with "noise". Noise is essential high frequency. Thus the output of this true differentiation circuit will be the desired derivative with the noise derivative superposed and by virtue of the fact that the special case of Eq. (8) is not usually met, the signal to noise ratio will be seriously reduced. Indeed, in many applications a true differentiator becomes a noise amplifier, e.g., when $f(t)$ is zero.

The circuit of Fig. 1b is designed to solve the implicit differential equation

$$(1 - a) \frac{dz}{dt} + z = \frac{dx}{dt} \tag{9}$$

and so to obtain an approximation to the derivative. For the special case when (a) is unity the actual derivative results. Clearly in this instance there can be no advantage in this circuit since it is more complex than that of Fig. 1a and gives an identical output. It is the case $a \neq 0$, which is of interest. The above equation is implemented on a computer by first integrating both sides of Eq. (9), resulting in

$$(1 - a)z + \int_0^t z dt = x \tag{10}$$

or

$$z = az - \int_0^t z dt + x \tag{11}$$

The transfer function of this circuit is

$$\frac{z}{x} = - \frac{S}{(1 - a)S + 1} \tag{12}$$

or with $\tau = (1 - a)$ we may write

$$\frac{z}{x} = - \frac{S}{(S\tau + 1)} \tag{13}$$

It is seen from Eq. (12) or (13) that when $a = 1$, the true derivative results, as we previously noted. Figure 2, shows the frequency response in this case, (line 1). For the general case $a \neq 1$. The frequency response is limited for circular frequencies greater than ω_1 defined by

$$\omega_1 = \frac{1}{\tau} = \frac{1}{(1 - a)} \tag{14}$$

This is illustrated in Fig. 2 (line 2). Thus, we readily appreciate that the introduction of the single pole causes input signals of frequency higher than ω_1 to pass with a constant amplification. Hence, we may achieve a considerable improvement in signal noise ratio in a particular circumstance by the correct positioning of the pole, i.e., by an appropriate choice of the parameter (a).

The transfer function for the circuit given in Fig. 1c is

$$\frac{e_0(s)}{e_{in}(s)} = - \frac{R_2C_1S}{(R_1C_1S + 1)(R_2C_2S + 1)} \tag{15}$$

For the special case in which R_1 and C_2 are zero and $R_2C_1 \neq 0$ this equation is identical to Eq. (1) and the discussion of circuit 1(a) is relevant. In the special case when either R_1 or C_2 is zero, the transfer function is similar to that for 1(b) and the behavior then follows the preceeding discussion. When these particular conditions are not met the transfer function contains the zero necessary to generate the derivative together with two arbitrarily located poles. For convenience, at this stage of our discussion, we shall put $R_1C_1 = R_2C_2 = \tau$. In this case Eq. (15) may be re-written as

$$\frac{e_0(S)}{e_{in}(S)} = - \frac{R_2C_1S}{(S\tau_1 + 1)^2} \tag{16}$$

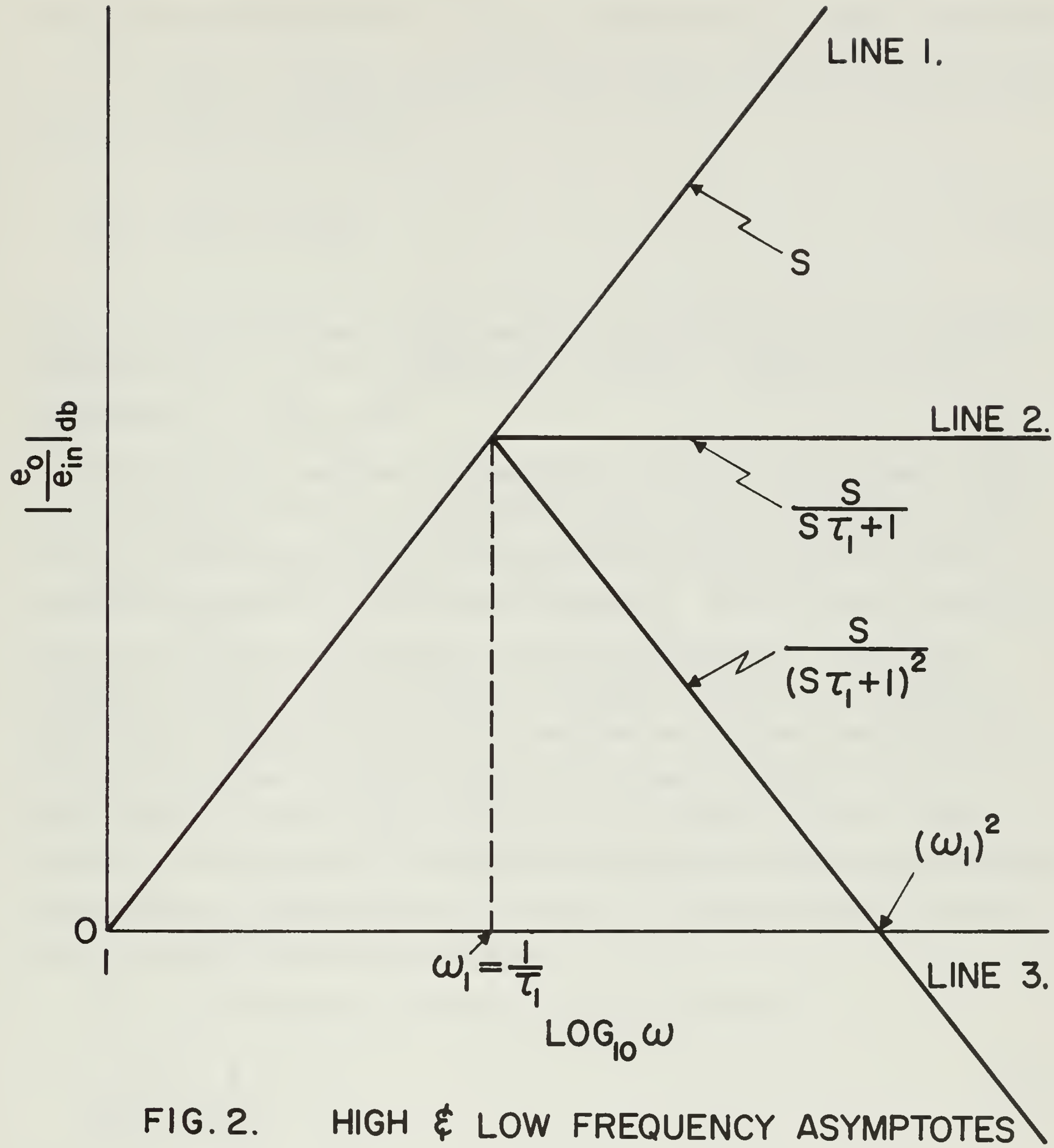

FIG. 2. HIGH & LOW FREQUENCY ASYMPTOTES OF $\left|\frac{e_o}{e_{in}}\right|_{db}$

The high and low frequency asymptotes of this transfer function are given in Fig. 2 (line 3). The circuit has the advantage of passing, with no amplification, frequencies of ω_1^2, and attenuating all higher frequencies. If, for design purposes, it is desired to operate upon signal frequencies up to ω_0 with a maximum error of 1 per cent, the time constant values should be chosen such that

$$R_1C_1 = R_2C_2 = \frac{1}{10\omega_0} \tag{17}$$

Since the error decreases approximately as the square of the frequency difference from ω_0, at a frequency of $\sqrt{10}\ \omega_0$ the error will be approximately 0.1 per cent.

The frequency response diagram given in Fig. 2 shows clearly that it is possible by the use of multiple poles to cause a derivative type circuit to discriminate against certain frequencies and yet maintain reasonable accuracy in a restricted bandwidth. Thus, the multiple pole system has distinct advantages in control applications. Generally speaking, a control signal consists of either a dc or low frequency signal and a superposed higher frequency noise. In most laboratories, the critical noise is 60cps. Thus, in a normal laboratory control application the circuit 1(c) finds the most common acceptance. When it is used it is customary to require that the 60 cycle noise passes without amplification. The location of the poles is then determined in accordance with the following proceedure.

At $\omega = 377$ rad/sec, the gain is zero db. Hence

$$\left|\frac{S}{(S\tau + 1)^2}\right|_{db} = 0 \tag{18}$$

or

$$\frac{\omega}{\sqrt{[1 - (\omega\tau)^2]^2 + (2\omega\tau)^2}} = 1 \tag{19}$$

therefore

$$[1 - (\omega\tau)^2]^2 + (2\omega\tau)^2 = \omega^2 \tag{20}$$

$$1 - 2(\omega\tau)^2 + (\omega\tau)^4 + 4(\omega\tau)^2 = \omega^2 \tag{21}$$

$$[1 + (\omega\tau)^2]^2 = \omega^2 \tag{22}$$

i.e.,

$$1 + (\omega\tau)^2 = \omega \tag{23}$$

Hence

$$\tau = \frac{1}{\sqrt{\omega}} = .0515 \text{ sec} \tag{24}$$

Thus

$$\omega_1 = \frac{1}{\tau_1} = 19.4 \text{ rad/sec} \tag{25}$$

It can be shown that if the corner frequency is fixed at 19.4 rad/sec a 1 per cent error exists at 1.94 rad/sec or 0.31 cps. Thus the circuit discriminates against the higher frequencies at the expense of accuracy in even the very low frequency spectrum. From Fig. 2 it is clear that with a two pole approximate derivative circuit it is preferable to make the poles coincident.

III. CIRCUITS BASED ON MATHEMATICAL APPROXIMATION TO THE DERIVATIVE

3.1. Feasability of Approach

Because of the practical problems experienced in derivative circuits it was decided that it might be profitable to investigate the feasibility of employing mathematical approximations in the construction of such circuits.

The derivative of $f(t)$ with respect to t is defined as

$$f'(t) = \lim_{\Delta t \to 0} \frac{(f_1 - f_2)}{\Delta t} \tag{26}$$

where f_1 is the value of $f(t)$ at time (t) and f_2 is value at time $(t - \Delta t)$. The ratio $(f_1 - f_2)/\Delta t$ approximates to the derivative when Δt is small, the smaller Δt the better the approximation. Physically, this fundamental definition can be interpreted in the following way:

The time derivative of a varying voltage is equal to a constant multiplied by the change in signal amplitude which occurs in a small but fixed interval of time.

We examine then the nature of this approximation to a derivative when the system is applied to a signal which consists of two components — a smooth signal and a superposed AC signal. If the time interval between the sample f_1 and the sample f_2 is correctly chosen, the AC components will be in phase, equal in magnitude and of the same polarity. Thus, when the signal samples are differenced the AC components will cancel each other out and $(f_1 - f_2)$ will be approximately proportional to the derivative of the smooth component. We have implied in this approach that the sampling period must be a function of the frequency of the AC signal. The reader will notice that there is a direct comparison between this approach and Lanczo's[5] differentiation technique for a Fourier Series. It is well recognized that if we differentiate a Fourier Series we usually get what is not wanted in the physical problem, namely, a large high frequency oscillation. This, of course, is precisely

what happens when we use a classic approach to obtain the derivative of a noisy signal. Lanczo introduced a practical approach to such a problem when the suggested

$$\left[f\left(t + \frac{\pi}{n}\right) - f\left(t - \frac{\pi}{n}\right)\right] \frac{n}{2\pi} \tag{27}$$

as an estimate of the derivative of $f(t)$ at the point t instead of the limit process which has dubious physical significance when we differentiate a Fourier Series approximation.

The question we must now answer is how can this simple approach be embodied in a computing circuit. There appears to be two distinct possibilities. First, we can sample the signal at discrete intervals of time and difference the samples. Alternatively, we can bifurcate the signal, delay one component relative to the other and then recombine the two parts through a summing network. The output of this network will be proportional to the derivative.

It seems apparent that the first approach is more suited to a digital system and the second to an analog method. It is, therefore, to the latter that we shall direct our attention. We begin by considering how this approximation fits into an electronic network and the first point which we clearly observe is that $1/\Delta t$ is the overall circuit gain. Thus, the system does not allow complete freedom in determining Δt but rather Δt must be chosen to be compatible with the characteristics of the amplifiers which are used. We must, of course, also consider the response characteristics we desire when assessing the circuit values. A cursory examination would indicate that only one time delay network is required to generate the potential difference which is proportional to the derivative. However, if the best discrimination against noise is required, this is not the case.

3.2. Two Pole, Two Leg Derivative Circuit

The analog circuit used is shown in Fig. 3. The transfer function of the circuit is, therefore

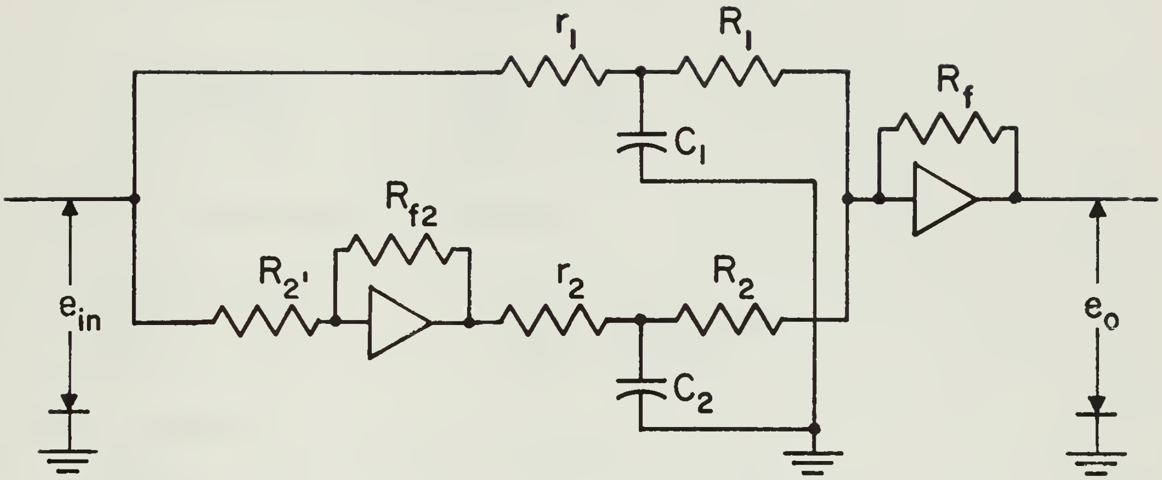

FIG. 3. TWO POLE, TWO LEG DERIVATIVE CIRCUIT

$$\frac{e_0(s)}{e_{in}(s)} = - \left\{ \frac{\frac{R_f}{r_1 + R_1}}{\left[S\left(\frac{C_1 r_1 R_1}{r_1 + R_1}\right) + 1\right]} - \frac{\frac{R_{f2}}{R_2'} \frac{R_f}{r_2 + R_2}}{\left[S\left(\frac{C_2 r_2 R_2}{r_2 + R_2}\right) + 1\right]} \right\} \tag{28}$$

By defining

$$\tau_1 = \frac{C_1 r_1 R_1}{r_1 + R_1} \quad , \quad \alpha_1 = \frac{R_f}{r_1 + R_1}$$

$$\tau_2 = \frac{C_2 r_2 R_2}{r_2 + R_2} \quad , \quad \alpha_2 = \frac{R_{f2}}{R_2'} \frac{R_f}{r_2 + R_2}$$

The above equation can be written as

$$\frac{e_0(s)}{e_{in}(s)} = -\alpha_1(\tau_2 - \tau_1) \quad \frac{S}{(S\tau_1 + 1)(S\tau_2 + 1)} \tag{29}$$

if the parameters are chosen such that

$$\alpha_1 = \alpha_2 \tag{30}$$

and

$$\tau_1 \neq \tau_2 \tag{31}$$

With the exception of the gain factor $\alpha_1(\tau_2 - \tau_1)$, which must be compensated for in the summing amplifier, the analysis of this circuit is similar to that for two pole approximation discussed previously. The difference is due to the fact that the two poles must be separated and hence the desired values of the corner frequencies can best be obtained by graphical methods using the asympotes of the frequency response gain curve.

Accuracy dictates that the poles should be closely spaced, but, the $(\tau_2 - \tau_1)$ gain factor, dictates their values. This approach demonstrates that mathematical approximations may have advantages.

3.3. The General Case

In a paper published in the Math Gazette[6], W. G. Bickley has given a set of tables for use in numerical computation of derivatives. In these tables coefficients are given for insertion into the general formula

$$\frac{A^m y_r^{(m)}}{m!} = \frac{1}{P!} \left(A_0 y_0 + A_1 y_1 + \dots A_p y_p\right) + \mathcal{E} \tag{32}$$

In these formulae, m = 1, 2, 3, etc., and reference to the order of the derivative of y with respect to x at the point $x = ra$ where $r = 0, 1, 2, \dots p$.

It is significant to note that in these general formulae the condition is always met.

$$\sum_{n=0}^{n=p} A_n = 0 \tag{33}$$

This general formula indicates that if we sample a signal at discrete times Δt, $2\Delta t$, etc., amplify these samples by an amount proportional to Bickley's coefficients and algebraically sum, the sum so derived is proportional to the n^{th} derivative. The number of time dealys (Δt's) used in the process determines the resulting number of poles generated in the overall transfer function of the circuit.

3.4. The Three Pole, Three Leg Derivative Circuit.

From Eq. (32), the formula for the three point approximation to a derivative is

$$y^1 \cong \frac{1}{2\Delta t} \left(A_1 y_1 + A_2 y_2 + A_3 y_3\right) \tag{34}$$

Figure 4 gives the analog implementation of the above equation. Setting

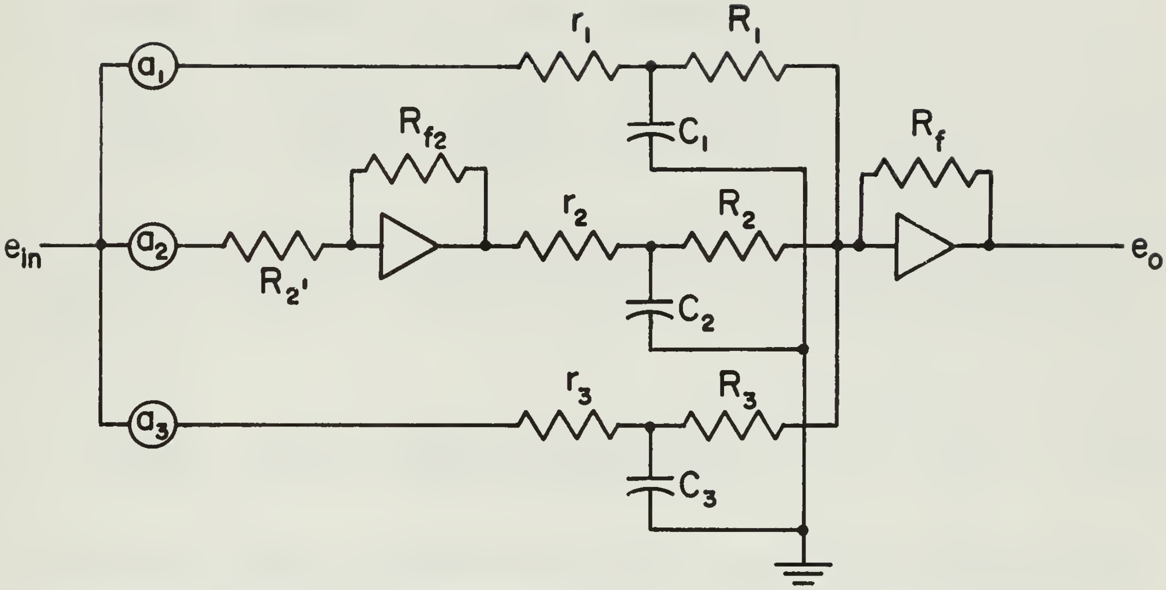

FIG. 4. THREE POLE, THREE LEG DERIVATIVE CIRCUIT

$$\tau_1 = \frac{C_1 r_1 R_1}{r_1 + R_1} \qquad \alpha_1 = \frac{R_f}{r_1 + R_1}$$

$$\tau_2 = \frac{C_2 r_2 R_2}{r_2 + R_2} \qquad \alpha_2 = \frac{R_{f_2}}{R_2'} \quad \frac{R_f}{r_2 + R_2}$$

$$\tau_3 = \frac{C_3 r_3 R_3}{r_3 + R_3} \qquad \alpha_3 = \frac{R_f}{r_3 + R_3}$$

the transfer function of the circuit becomes

$$\frac{e_0(s)}{e_{in}(s)} = - \left[\frac{a_1\alpha_1}{S\tau_1 + 1} - \frac{a_2\alpha_2}{S\tau_2 + 1} + \frac{a_3\alpha_3}{S\tau_3 + 1} \right] \tag{35}$$

Based on the selection of the time constant such that $\tau_n = n\tau$ as suggested by Bickley's constant interval, the transfer function can be written as

$$\frac{e_0(s)}{e_{in}(s)} = 2a_1\alpha_1\tau_1 \left[\frac{S}{(S\tau_1 + 1)(S\tau_2 + 1)} \right] \tag{36}$$

provided the circuit parameter are chosen such that the following conditions are satisfied:

$$a_1\alpha_1 - a_2\alpha_2 + a_3\alpha_3 = 0 \tag{37}$$

$$6a_1\alpha_1 - 3a_2\alpha_2 + 2a_3\alpha_3 = 0 \tag{38}$$

$$5a_1\alpha_1 - 4a_2\alpha_2 + 3a_3\alpha_3 \neq 0 \tag{39}$$

The ratio that satisfies the above three equations was found to be 1, -4, 3. This ratio is identical to the coefficients listed by Bickley for his three point differentiation formula.

Thus, a three-pole approximation has been generated using the mathematical finite difference method. The value of three poles over two is obvious, they permit greater accuracy in the lower frequency range. The net effect of three poles can be a -12 db/octave slope through the higher frequency chosen to have 0 db gain.

Figure 5 shows the theoretical frequency response of the three-pole approximation derivative circuit given in Fig. 4. The time constants used were

$$\tau_1 = .011, \quad \tau_2 = .022, \quad \text{and} \quad \tau_3 = .033 \quad .$$

The estimated response of the double-pole circuit with equal time constants is included for comparison. The curves illustrate clearly that the three-pole circuit has considerably more accuracy in the lower frequency region, although both circuits pass 60 cps without gain.

3.5. The Four-Pole, Four-Leg Derivative Circuit

The improvement of the low-frequency response, resulting from the inclusion of an additional pole leads one to consider the practicality of a four-pole derivative network. Following the procedure used in the previous sections, the desired derivative is expressed in the form

$$y' \cong K(A_1 y_1 + A_2 y_2 + A_3 y_3 + A_4 y_4) \tag{40}$$

Figure 6 shows the analog implementation of this equation. Setting

$$\tau_1 = \frac{r_1 R_1 C_1}{r_1 + R_1} \qquad \alpha_1 = \frac{R_f}{r_1 + R_1}$$

$$\tau_2 = \frac{r_2 R_2 C_2}{r_2 + R_2} \qquad \alpha_2 = \frac{R_{f_2}}{R_2'} \quad \frac{R_f}{r_2 + R_2}$$

$$\tau_3 = \frac{r_3 R_3 C_3}{r_3 + R_3} \qquad \alpha_3 = \frac{R_f}{r_3 + R_3}$$

$$\tau_4 = \frac{r_4 R_4 C_4}{r_4 + R_4} \qquad \alpha_4 = \frac{R_{f_4}}{R_4'} \quad \frac{R_f}{r_4 + R_4}$$

the transfer function of the above circuit becomes

$$\frac{e_0(s)}{e_{in}(s)} = - \left[\frac{a_1 \alpha_1}{S\tau_1 + 1} - \frac{a_2 \alpha_2}{S\tau_2 + 1} + \frac{a_3 \alpha_3}{S\tau_3 + 1} - \frac{a_4 \alpha_4}{S\tau_4 + 1} \right] \tag{41}$$

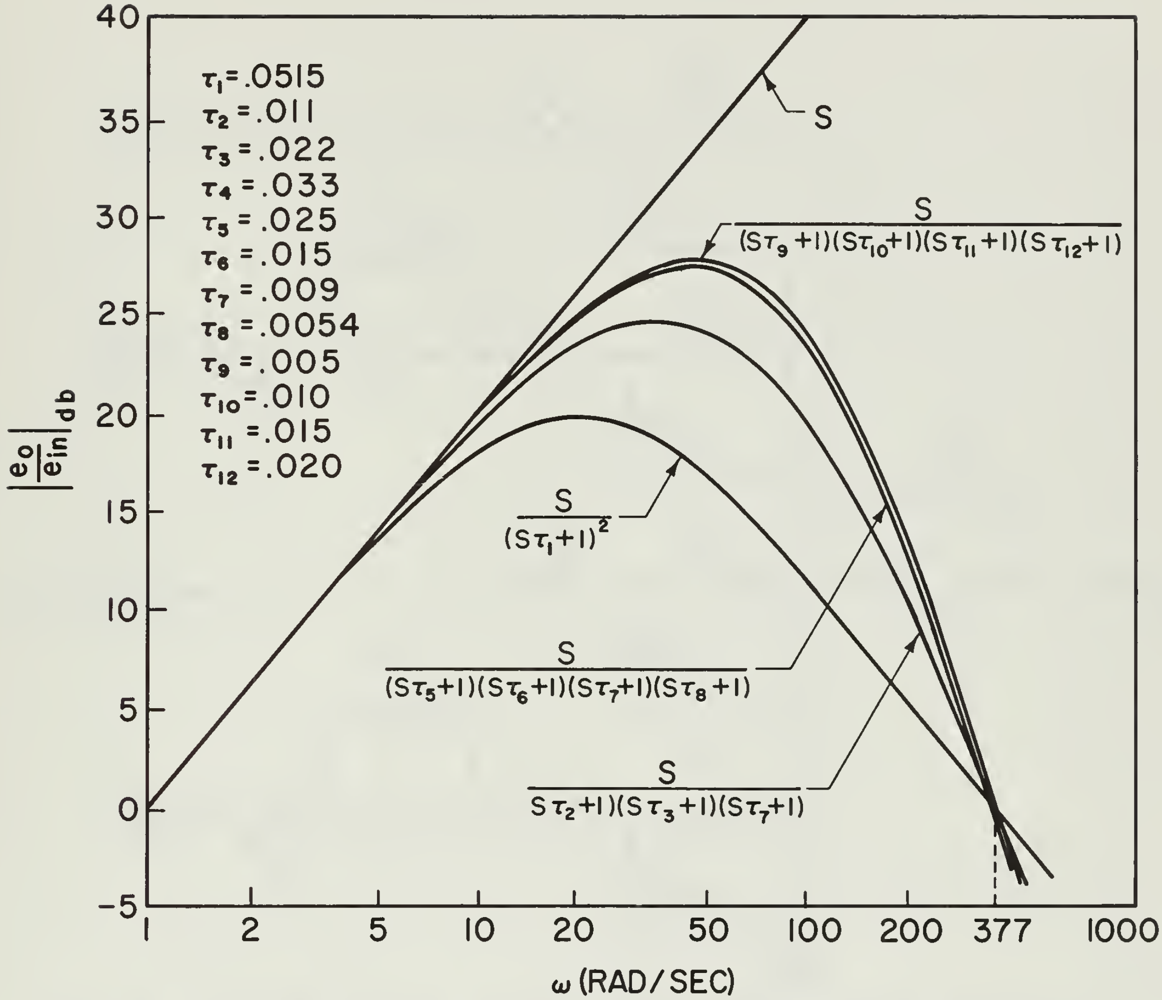

FIG. 5 COMPARISON OF FREQUEY RESPONSE FOR THE VARIOUS DERIVATIVE SYSTEMS.

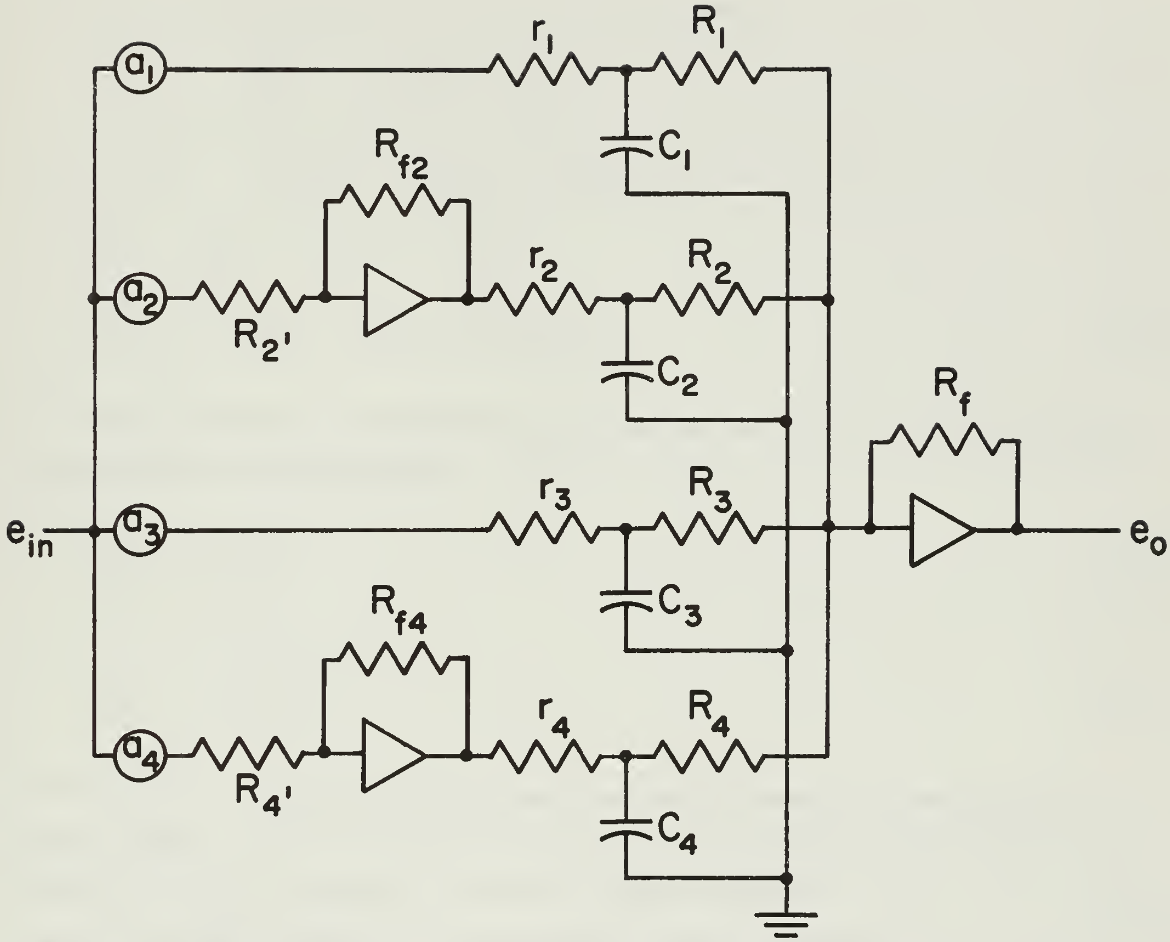

FIG. 6. FOUR POLE, FOUR LEG DERIVATIVE CIRCUIT

Again chosing the time constants such that $\tau_n = n\tau (n = 1,2,3,4)$, the transfer function becomes

$$\frac{e_0(s)}{e_{in}(s)} = -6a_1\alpha_1\tau_1 \left[\frac{S}{(S\tau_1 + 1)(S\tau_2 + 1)(S\tau_3 + 1)(S\tau_4 + 1)} \right] \tag{42}$$

if the circuit parameters are chosen such that

$$12a_1\alpha_1 - 6a_2\alpha_2 + 4a_3\alpha_3 - 3a_4\alpha_4 = 0 \tag{43}$$

$$26a_1\alpha_1 - 19a_2\alpha_2 + 14a_3\alpha_3 - 11a_4\alpha_4 = 0 \tag{44}$$

$$a_1\alpha_1 - a_2\alpha_2 + a_3\alpha_3 - a_4\alpha_4 = 0 \tag{45}$$

$$9a_1\alpha_1 - 8a_2\alpha_2 + 7a_3\alpha_3 - 6a_4\alpha_4 \neq 0 \tag{46}$$

The solution of the above equations yields the following relationship between the parameters

$$a_2\alpha_2 = 12a_1\alpha_1 \tag{47}$$

$$a_3\alpha_3 = 27a_1\alpha_1 \tag{48}$$

$$a_4\alpha_4 = 16\ a_1\alpha_1 \tag{49}$$

At this point it must be noted that the ratio among the parameters as given by Eqs. (47)-(49), is not the same as the coefficients listed by Bickley for the four-point derivative approximation formula. Investigation showed that upon using the coefficients listed by Bickley, the resulting transfer function contained an additional zero. This zero was of the form $(S\tau^2 + 1)$, thus it would cause little change in the low-frequency response of the circuit.

While Eq. (42) gives a satisfactory form for a derivative approximation, the gain requirement dictated by the $6a_1\alpha_1\tau_1$ coefficient requires further consideration. To achieve an exact derivative in the low frequency range this coefficient must equal unity. The a_n's represent potentiometer settings, and therefore must have values of unity or less.

Since the $a_n\alpha_n$'s represent the relative gains of the four legs of the circuit, the gain may be written as

$$6\left(\frac{a_3\alpha_3}{27}\right)\tau_1 = 1 \tag{50}$$

because

$$a_3\alpha_3 = 27\, a_1\alpha_1 \tag{48}$$

In order to discriminate against 60 cps, as with the other circuits discussed, a value of $\tau_1 = .005$ sec. was found convenient. Hence from Eq. (50):

$$\alpha_3 = \frac{900}{a_3} \tag{51}$$

which, since

$$a_3 \leq 1$$

gives

$$\alpha_3 \geq 900 \tag{52}$$

But α_3 was defined as

$$\frac{R_f}{r_3 + R_3}$$

and therefore

$$R_f \geq 900(r_3 + R_3) \tag{53}$$

Hence it follows that the overall system gain must be greater than 900. A gain of this magnitude makes such a circuit impractical for operations where economy dictates simple, unsophisticated operational amplifiers.

The theoretical gain curve for this network is shown in Fig. 5. As expected, there is greater accuracy at low frequency than in the two- or three-pole cases previously discussed.

3.6. The Four-Pole, Two-Leg Derivative Circuit.

The impracticability of the previously described four-pole system, arising from the high inherent amplification requirement, caused consideration to be given to alternate approaches. The first of these is shown in Fig. 7. If for this circuit we set

$$\tau_1 = C_1 r_1 R_1$$

$$\tau_2 = \frac{C_1 r_1 R_1}{2r_1 + R_1} \qquad \alpha_1 = \frac{R_f}{2r_1 + R_1}$$

$$\tau_3 = C_2 r_2 R_2 \qquad \alpha_2 = \frac{R_f}{2r_2 + R_2}$$

$$\tau_4 = \frac{C_2 r_2 R_2}{2r_2 + R_2}$$

Then the resulting transfer function is given by

$$\frac{e_0(s)}{e_{in}(s)} = -\left[\frac{a_1\alpha_1}{(S\tau_1 + 1)(S\tau_2 + 1)} - \frac{a_2\alpha_2}{(S\tau_3 + 1)(S\tau_4 + 1)}\right] \tag{54}$$

This tranfer function may be written as

$$\frac{e_0(s)}{e_{in}(s)} = -a_1\alpha_1[(\tau_3 + \tau_4) - (\tau_1 + \tau_2)] \left[\frac{S}{(S\tau_1 + 1)(S\tau_2 + 1)(S\tau_3 + 1)(S\tau_4 + 1)}\right] \tag{55}$$

if the parameters are chosen such that the following equations are satisfied:

$$a_1\alpha_1 - a_2\alpha_2 = 0 \tag{56}$$

$$\tau_1\tau_2 - \tau_3\tau_4 = 0 \tag{57}$$

$$(\tau_1 + \tau_2) - (\tau_3 + \tau_4) \neq 0 \tag{58}$$

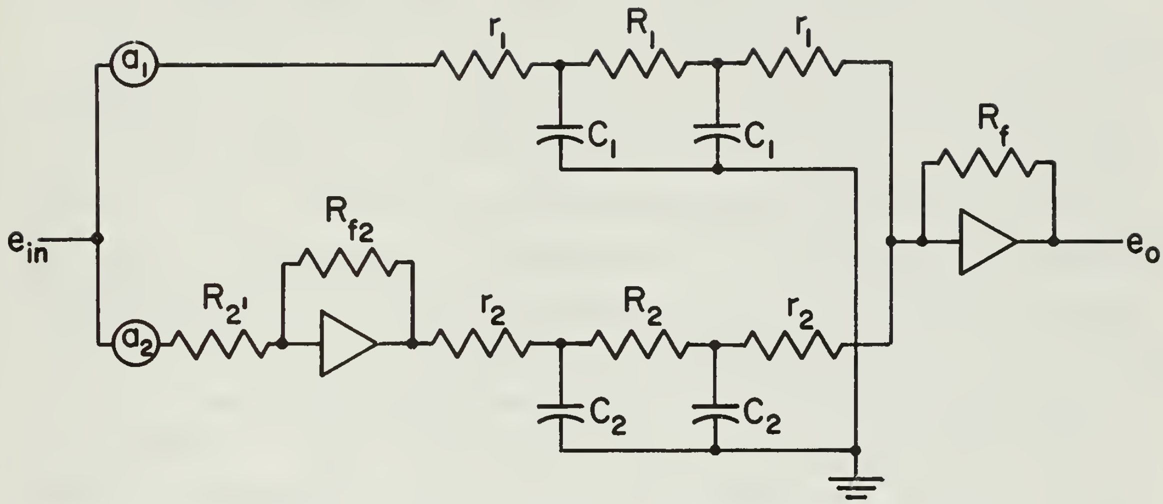

FIG. 7. FOUR POLE, TWO LEG DERIVATIVE CIRCUIT

As in the previous derivative approximations, we wish to choose the time constants of the poles such that 60 cps is passed without amplification. This requirement provides another equation relating the τ 's , i.e.,

$$\log 377 \cong \log 377\tau_1 + \log 377\tau_2 + \log 377\tau_3 + \log 377\tau_4 \qquad (59)$$

but it clearly does not give a unique solution. The optimum approach appears to be to choose two of the τ values and determine the other two from these, graphically. Figure 8 shows the proceedure adopted.

The graphics of Fig. 8 were derived from the high- and low-frequency asympotes of the transfer function given in Eq. (55). The procedure was: First construct the four-equipole envelope with $\omega = 1$ at zero db and $\omega = 377$ at zero db as the predetermined vertices (the lines OA and OB have slopes of +6db/octave and -18db/octave respectively). Second, construct the double pair envelope. This was done by choosing point C on line OA at a position corresponding to $\omega_1 = 40$. The line CD then drawn with the required slope of -6db/octave, hence point D was determined. Next, through point C, a line with zero slope was drawn and through point D a line with -12db/octave slope. The two undetermined poles of the four unequal pole system are then the intercepts of any line parallel to CD on the lines CE and ED.

This construction method is completely general as it is merely a graphical solution to Eqs. (57) and (59). The values obtained from the example given in Fig. 8 are:

$\tau_1 = .025$ $\qquad \tau_3 = .015$

$\tau_2 = .0054$ $\qquad \tau_4 = .009$

These values satisfy all the requirements of Eqs. (57), (58), and (59) and yield the following transfer function for the circuit given in Fig. 7.

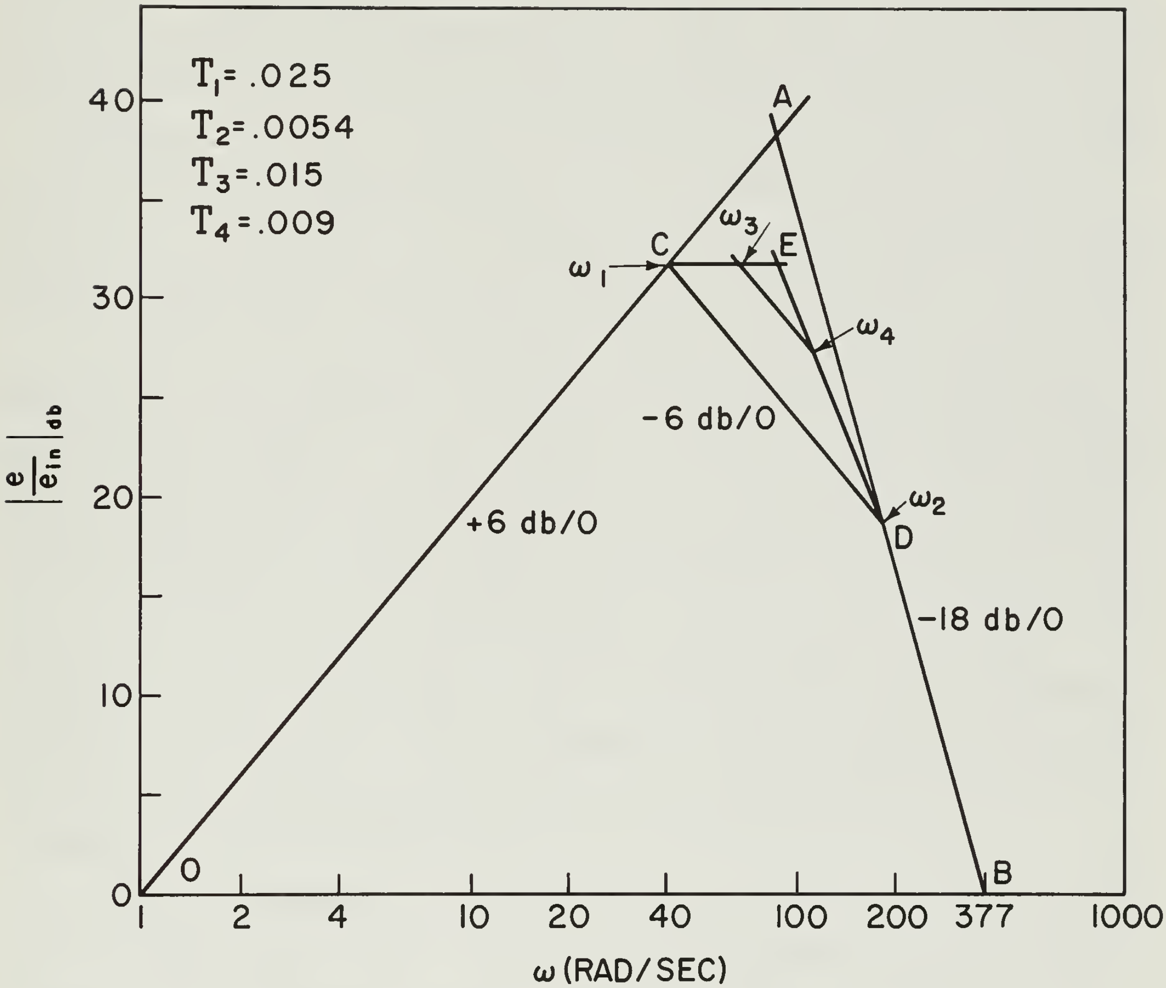

FIG. 8. METHOD OF DETERMINING POLE LOCATIONS

$$\frac{e_0(s)}{e_{in}(s)} = -a_1\alpha_1(.0054)\left[\frac{S}{(S\tau_1+1)(S\tau_2+1)(S\tau_3+1)(S\tau_4+1)}\right] \tag{60}$$

Since a_1 is a potentiometer setting, and is therefore less than unity, for the correct magnitude of the derivative at low frequencies, α_1 must be equal to or larger than 185. This gain is much more practical than the 900 obtained from the previous four-pole approximation circuit.

With the construction method outlined in Fig. 8, however, the gain requirement can be adjusted since it is in effect the reciprocal of $[(\tau_3 + \tau_4) - (\tau_1 + \tau_2)]$. The theoretical frequency response of this network is shown in Fig. 5 for comparison with those previously discussed.

All circuits have been examined in the laboratory, using a Donner Model 3500 Analog Computer, as part of this program. Our experiments show that the output of each circuit for various control type inputs and the frequency response characteristics are in agreement with our theoretical predictions. The results of the frequency response tests did not vary significantly from those shown in Fig. 5. The response of the two-leg, four-pole derivative device, shown in Fig. 7, to a range of inputs is given in Figs. 9-13. Results of the other circuits showed no marked deviation and so are not included. Figures 9-13 show the input signals, the derivative of the input and the integrated derivative for comparison with the input signal.

Following the method of synthesis used in the last circuit, several other circuits were devised using various combinations of R-C time delays. These circuits, along with their respective transfer functions are given in Appendix A. The analysis of each circuit follows the format previously described.

3.7. An Integral Approximation to the Derivative.

In Fig. 14 we show a comparison between a Fourier series, a Fejer sum and Lanczo's α factor curves for a 12 term approximation to a rectangular wave. These curves indicate that the Lanczo's α factor method has a clear advantage. Although we shall not present the argument here, we point out that the Lanczo's α method is essentially the same

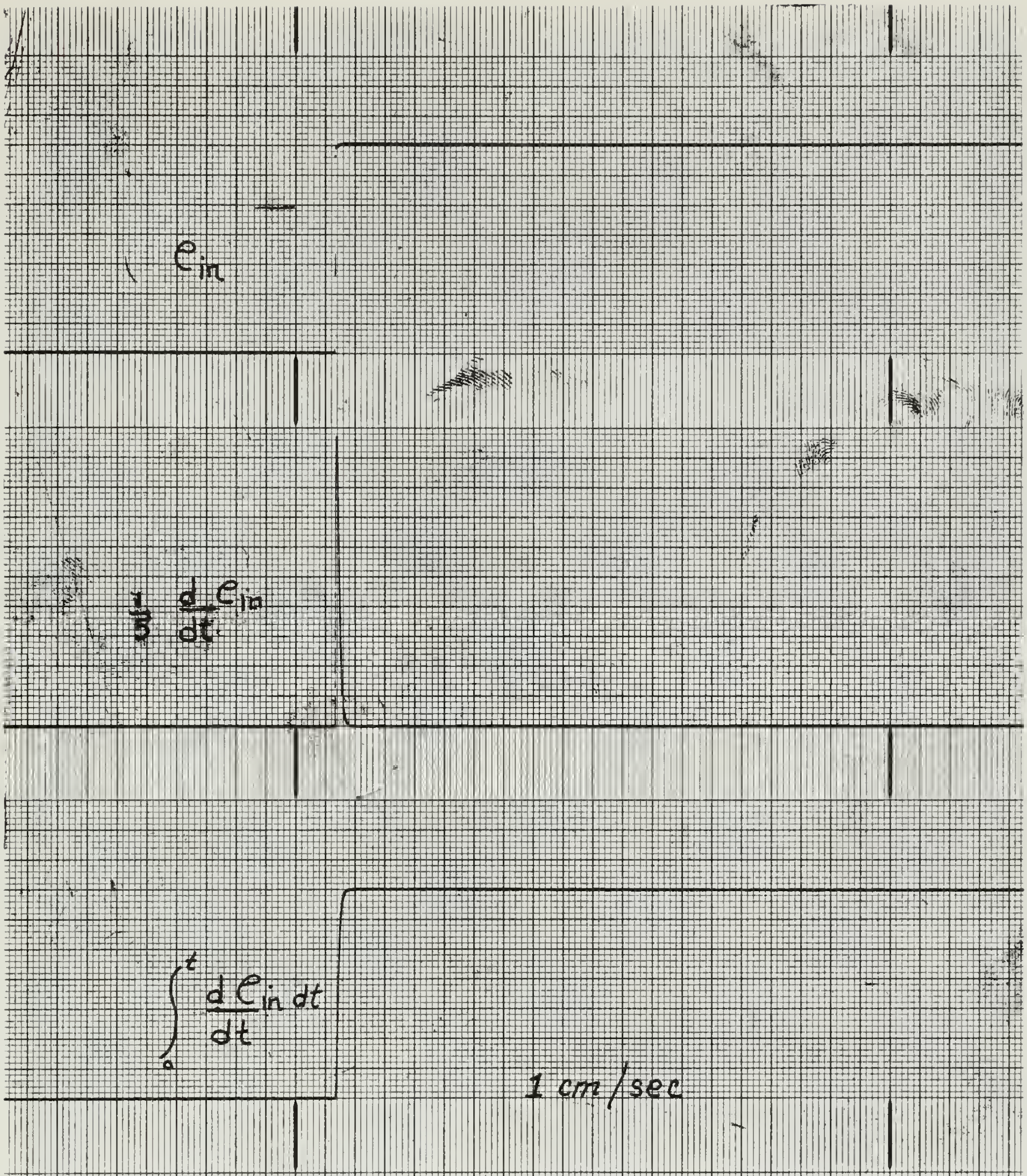

FIG. 9

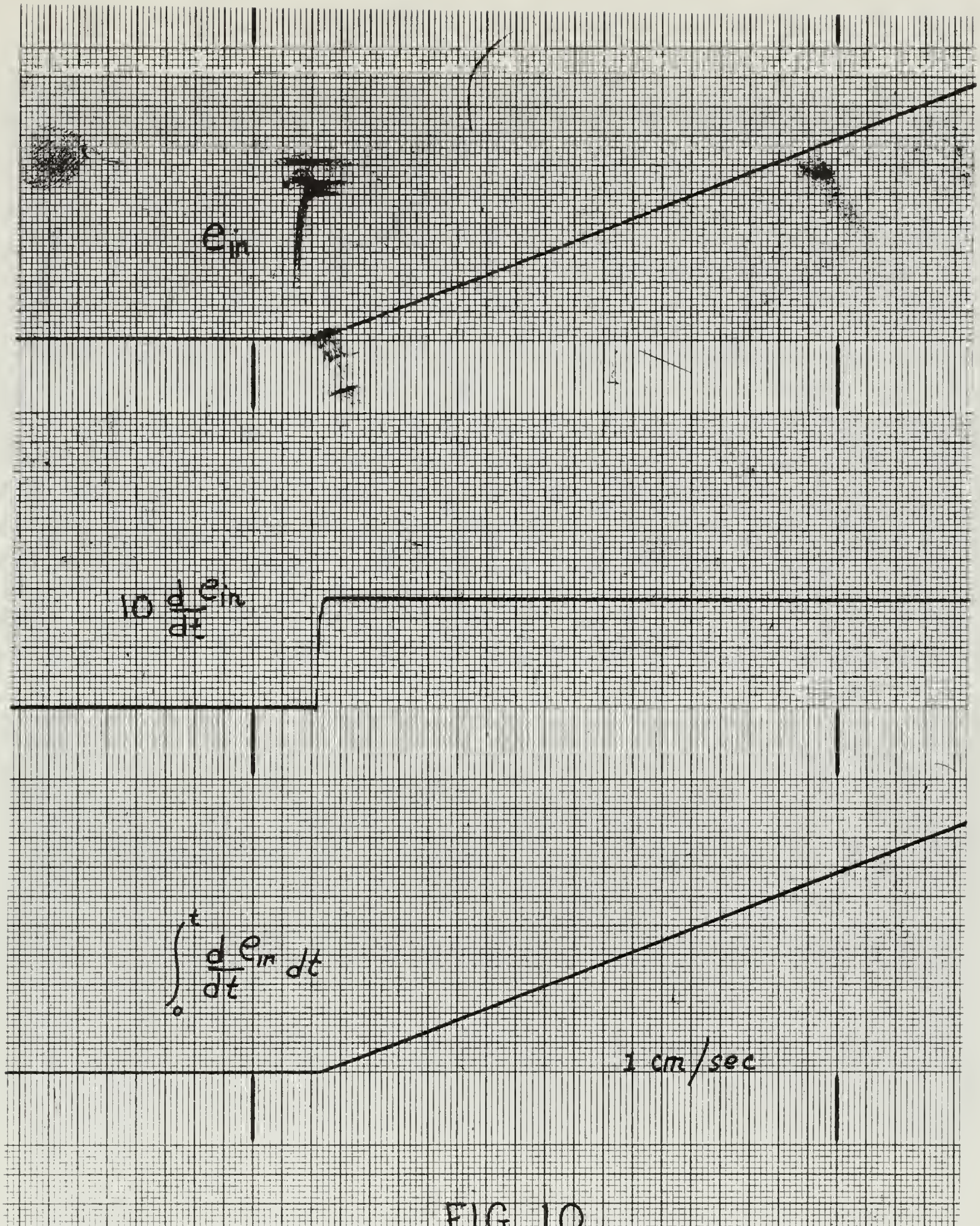

FIG. 10

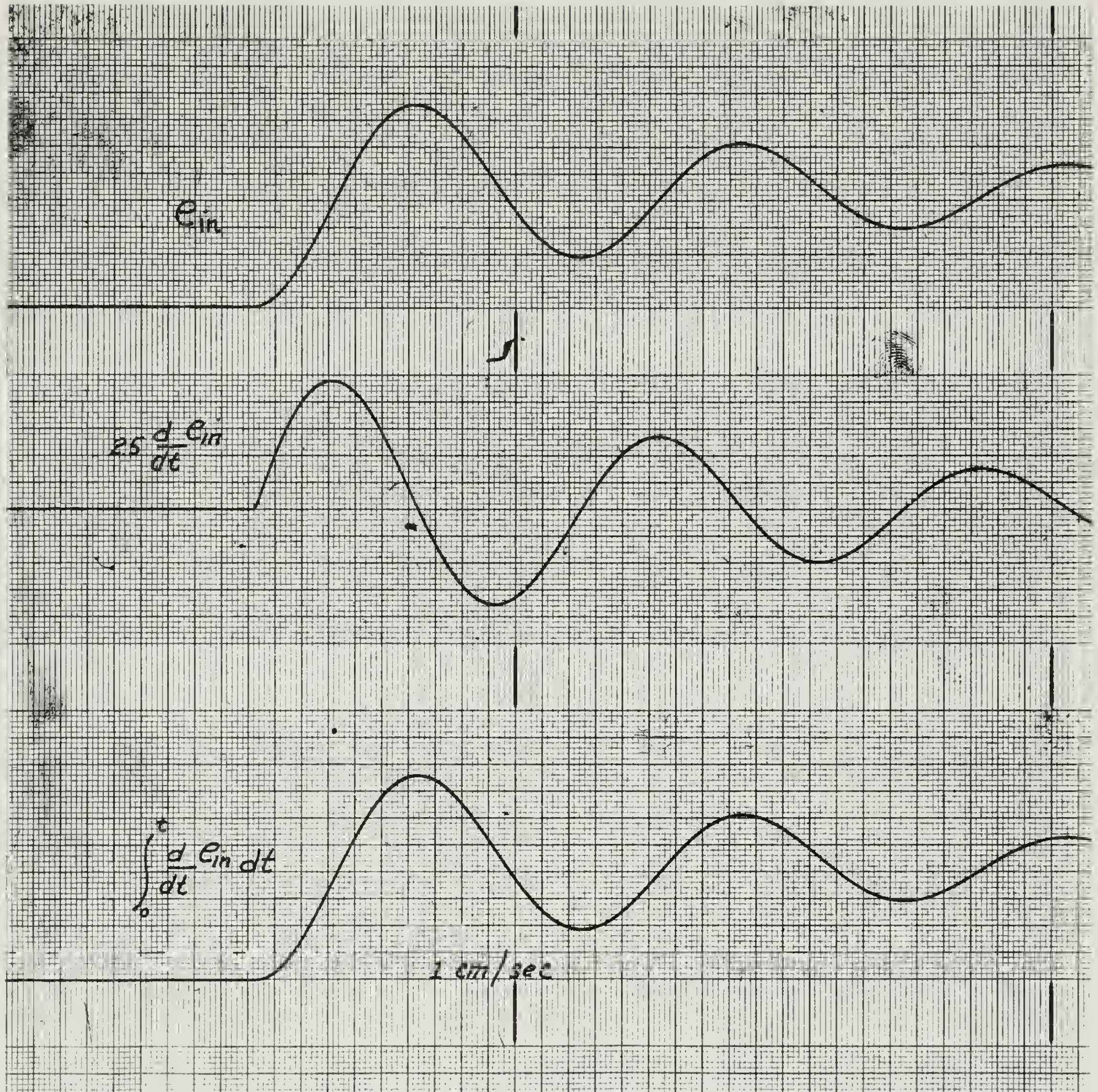

FIG. 11

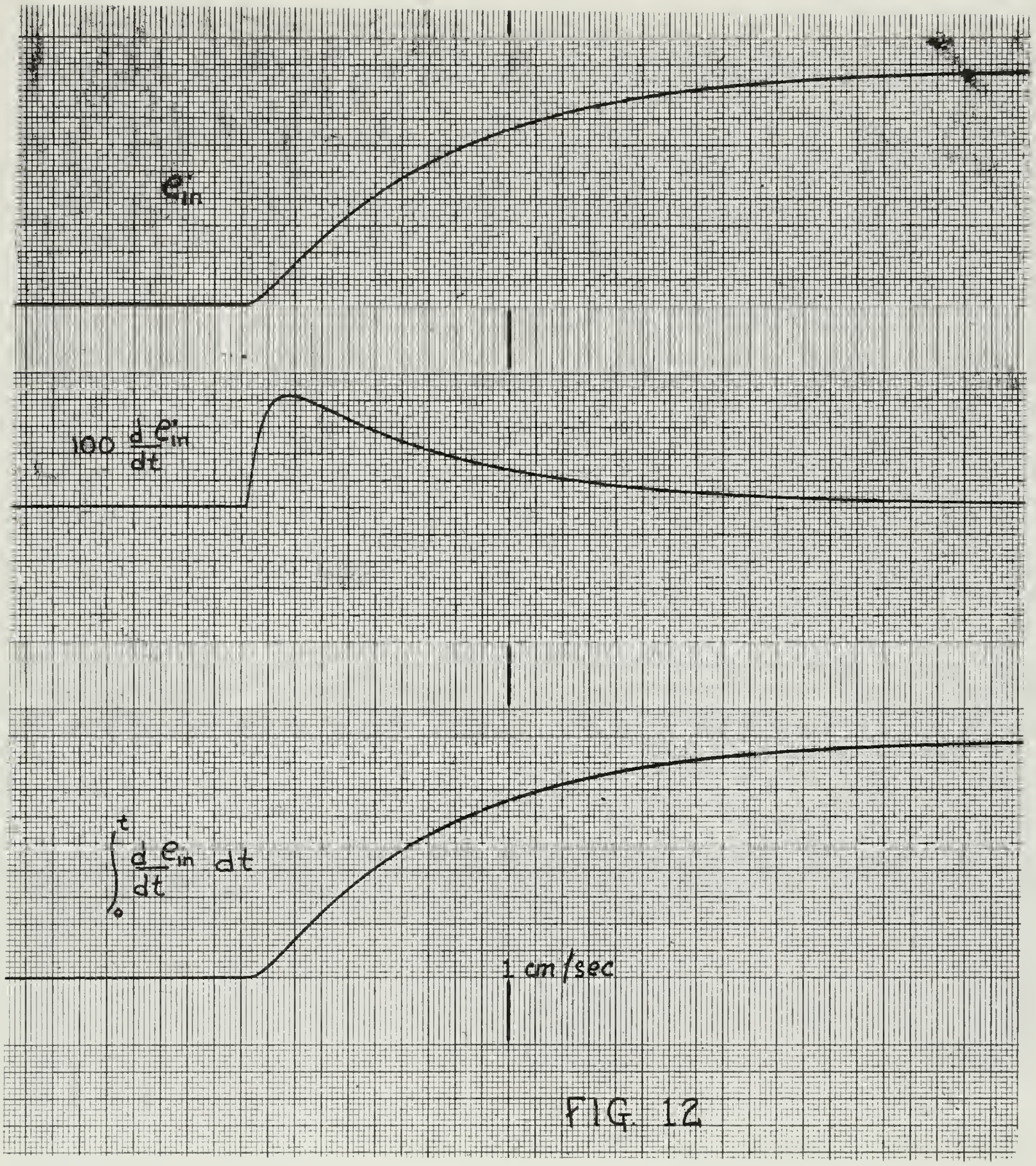

FIG. 12

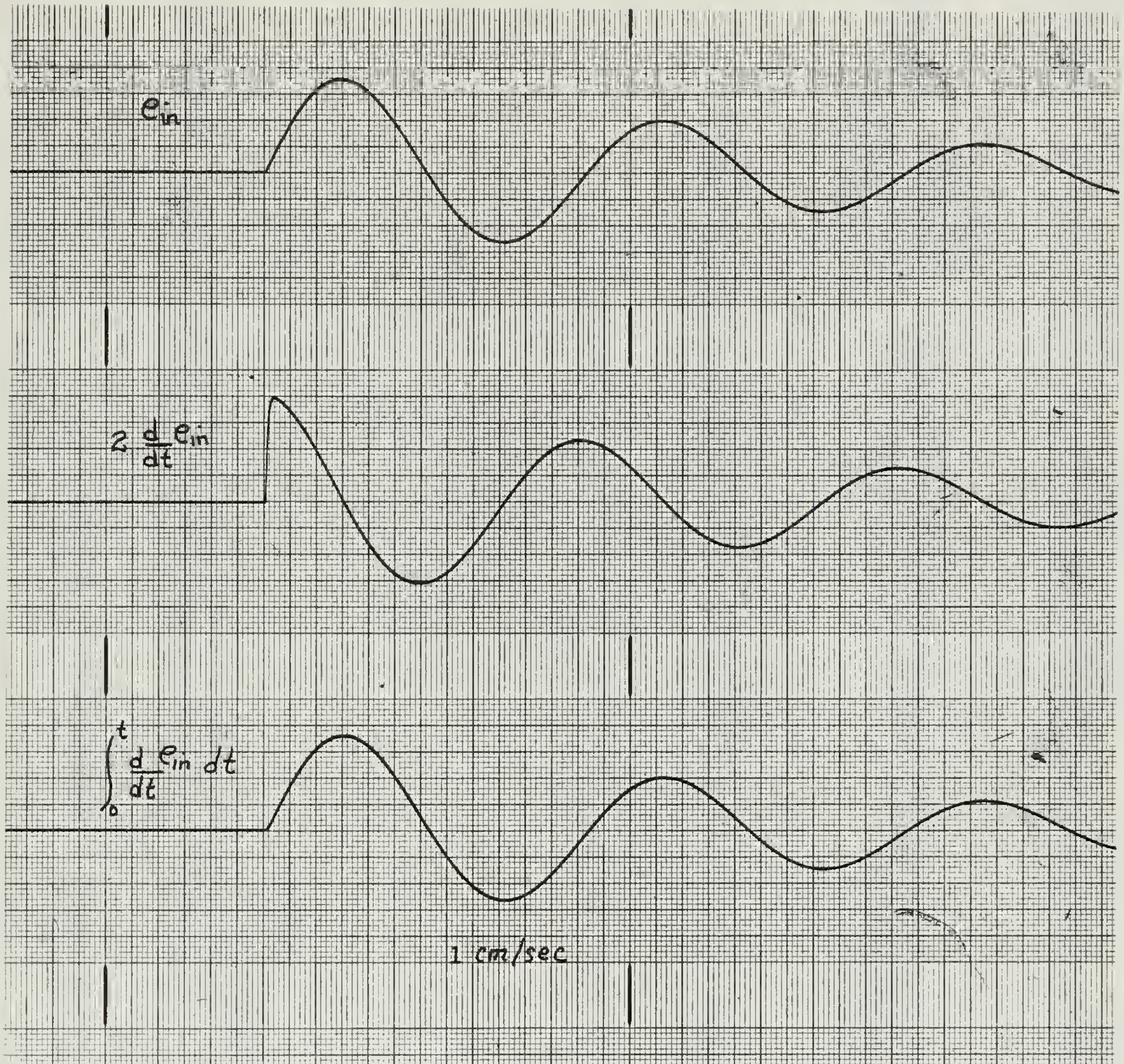

FIG. 13

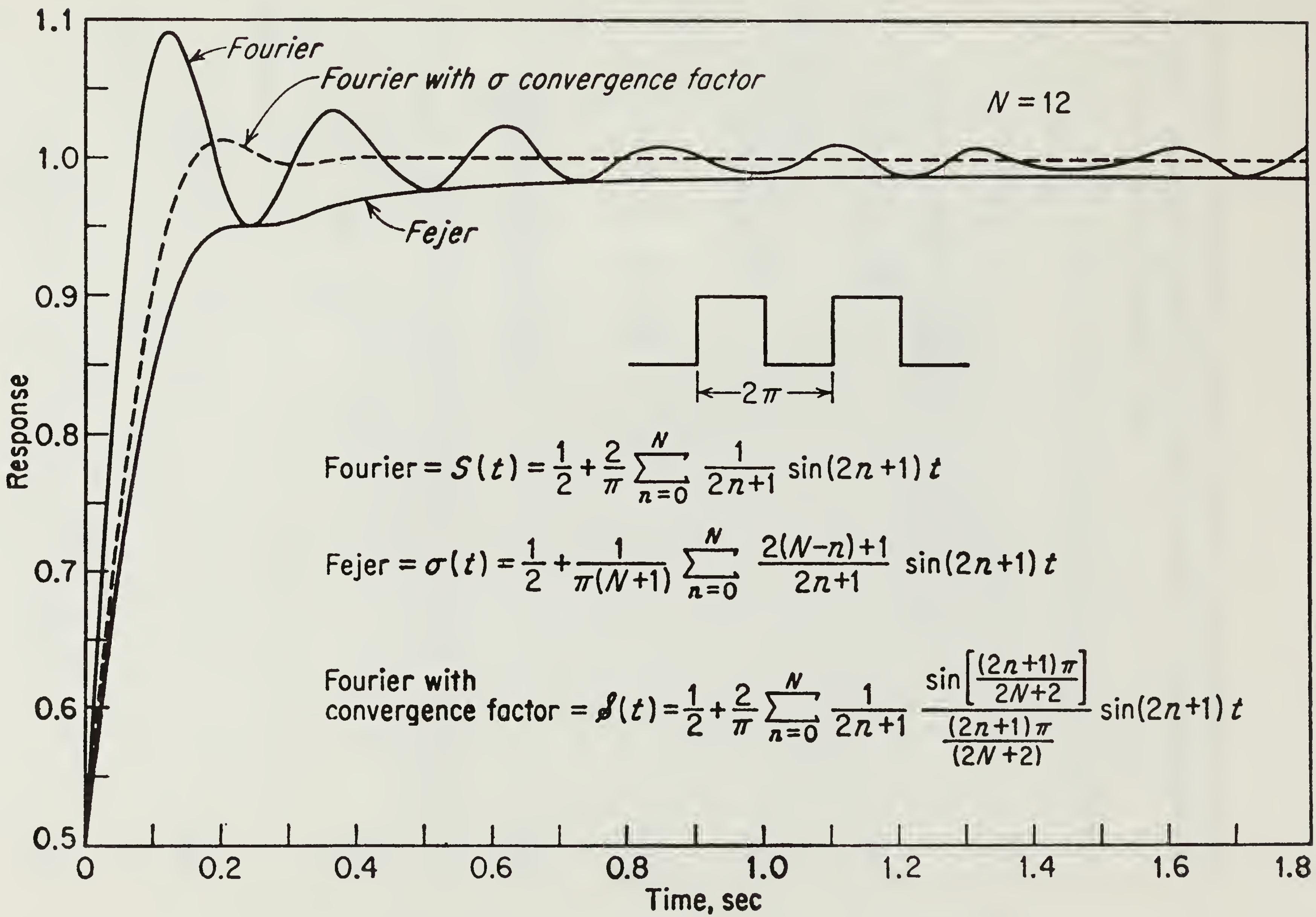
Fourier
Fourier with σ convergence factor
N = 12
Fejer
2π
Fourier = S(t) = 1/2 + 2/π Σ_{n=0}^{N} 1/(2n+1) sin(2n+1) t
Fejer = σ(t) = 1/2 + 1/(π(N+1)) Σ_{n=0}^{N} (2(N−n)+1)/(2n+1) sin(2n+1) t
Fourier with convergence factor = s(t) = 1/2 + 2/π Σ_{n=0}^{N} 1/(2n+1) sin[(2n+1)π/(2N+2)] / ((2n+1)π/(2N+2)) sin(2n+1) t
Response
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
Time, sec

FIG. 14

approach as the derivative approach to which we have previously referred. It seems reasonable, however, to consider whether or not there may be a possible analogy with the Fejer sum.

If we refer to Fig. 15 then we see that the derivative is approximately given by

$$(f_1 - f_3)/2\Delta t \tag{61}$$

Let us introduce the median f_2. Then the area

$$A_1 \cong (f_3 + f_2)\ \frac{\Delta t}{2} \tag{62}$$

and the area

$$A_2 \cong (f_2 + f_1)\ \frac{\Delta t}{2} \tag{63}$$

Thus

$$A_2 - A_1 \cong (f_1 - f_3)\ \frac{\Delta t}{2} \tag{64}$$

or

$$\frac{(A_2 - A_1)}{\Delta t^2} \cong \frac{(f_1 - f_3)}{2\Delta t} \cong \frac{df}{dt} \tag{65}$$

Now the area A_1 can be written as the difference of two integrals, i.e.,

$$A_1 = \int_0^{t-\Delta t} f(t)dt - \int_0^{t-2\Delta t} f(t)dt \tag{66}$$

Likewise

$$A_2 = \int_0^{t} f(t)dt - \int_0^{t-\Delta t} f(t)dt \tag{67}$$

Therefore

$$A_2 - A_1 = \int_0^{t} f(t)dt - 2\int_0^{t-\Delta t} f(t)dt + \int_0^{t-2\Delta t} f(t)dt \tag{68}$$

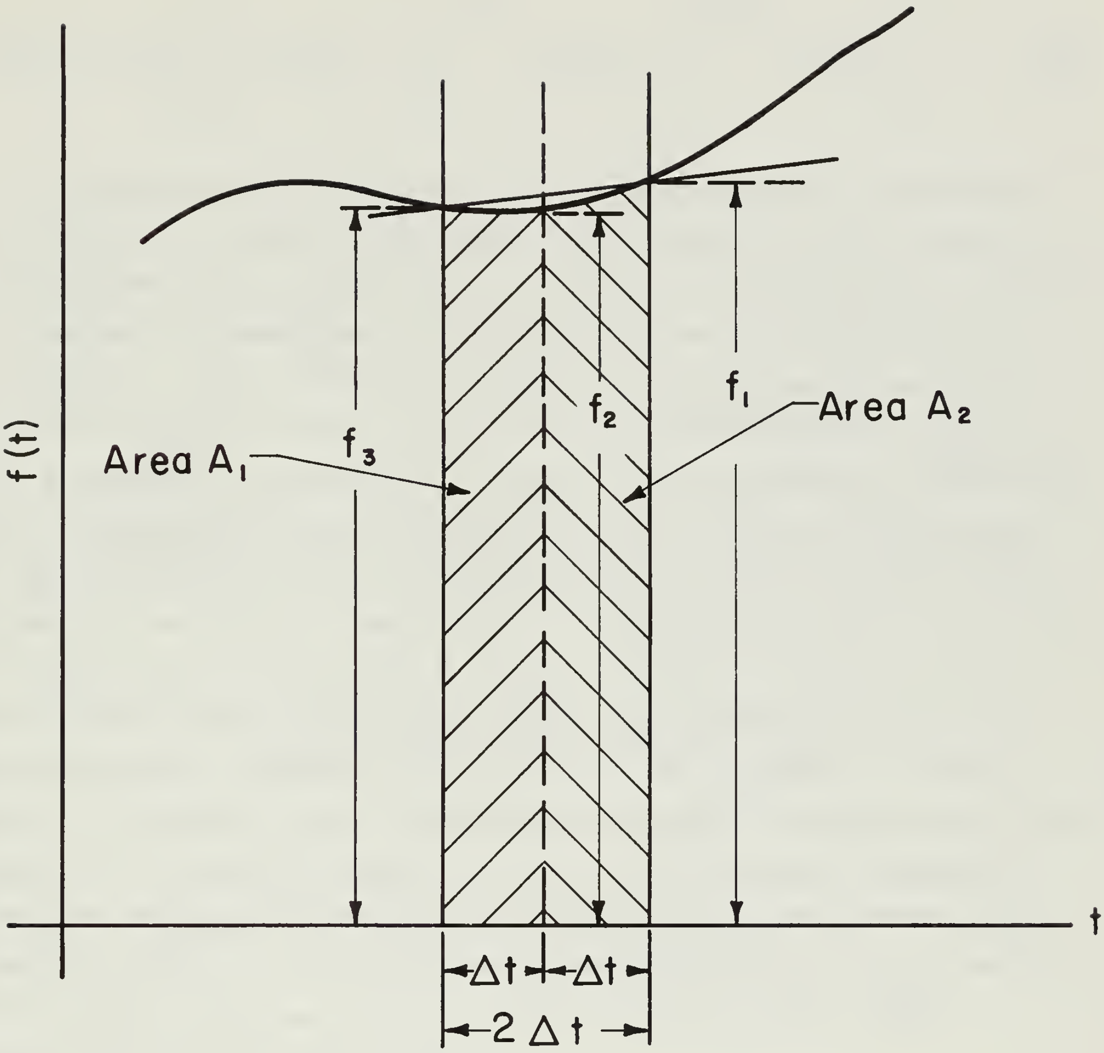

FIG. 15. AREA APPROXIMATION TO THE DERIVATIVE

Thus it follows from Eq. (68) that the derivative df/dt can be expressed in terms of difference in integrals. Thus

$$\frac{df}{dt} \cong (\Delta t)^2 \left[\int_0^t f(t)dt - 2 \int_0^{t-\Delta t} f(t)dt + \int_0^{t-2\Delta t} f(t)dt \right] \tag{69}$$

We examine now the nature of this approximation to a derivative when the system is applied to a signal which consists of two components. A smooth signal and a superposed AC signal. We see immediately that if Δt is correctly chosen then the integrals of the AC components are all identical. Thus, since the algebraic sum of the gains associated with these integrals is zero, they will be self-compensating so far as AC is concerned. We observe, too, that if this mathematical expression is to be transformed into an electronic circuit then the input signal must be trifurcated and the three arms delayed relative to each other. Δt takes on two special significances. First, it is equal to the relative time delay between adjacent arms and secondly the inverse of its square is the overall circuit gain. Thus, as before the system does not allow complete freedom in the choice of Δt, instead Δt must be chosen to be compatible with the characteristics of the amplifiers which are used. The integrating amplifiers act as very definite smoothing devices for cyclic noise and the delay circuits are very powerful peak or impulse filters. As in the other approximation circuits the smaller the Δt which we can use the more nearly will the output signal approach the true derivative level.

3.8. Integral Difference Derivative Circuit.

Figure 16 shows the analog network used to implement Eq. (69). The practicality of this circuit is limited, however, because of the tendency of the input integrating amplifier to saturate for any input of constant polarity. Analysis of the transfer function for the network given in Fig. 16 shows that the amplifiers one and three may be interchanged without effecting the function. Thus, the circuit of Fig. 17 gives the same result as that of Fig. 16, but prevents saturation. The transfer function of this circuit is:

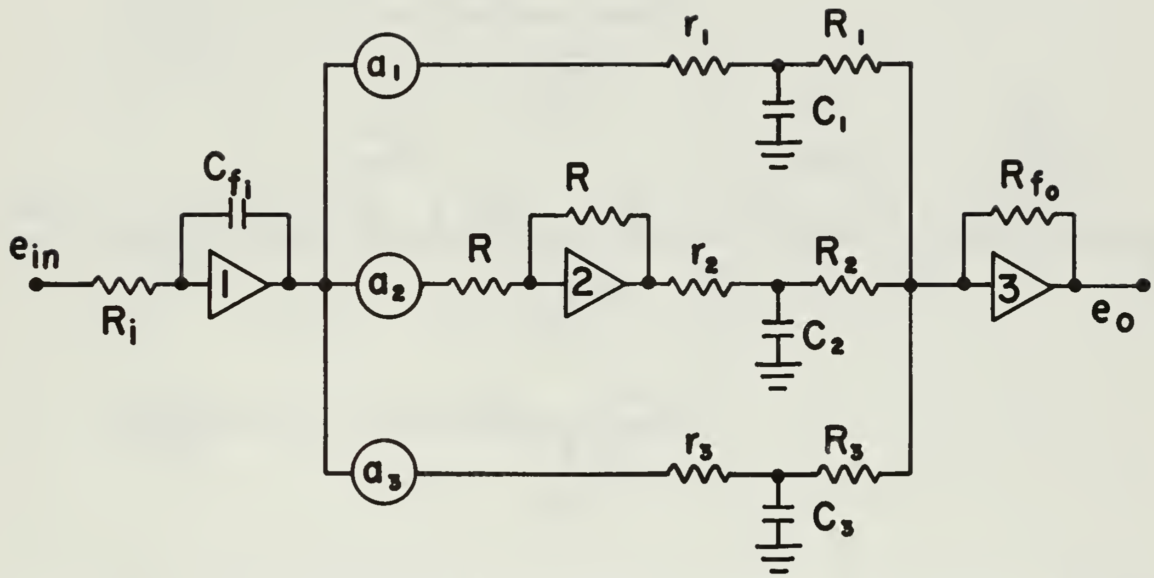

FIG. 16. THREE POLE DERIVATIVE CIRCUIT.

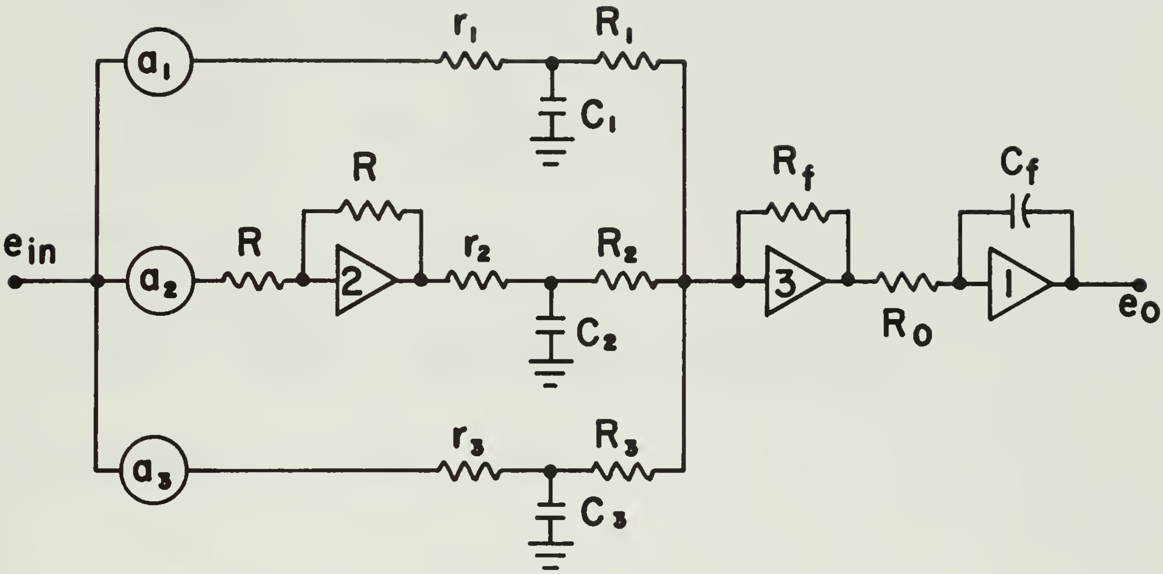

FIG. 17. THREE POLE CIRCUIT GIVING 1ST AND 2ND DERIVATIVES.

$$\frac{e_0(s)}{e_{in}(s)} = - \frac{1}{SR_0C_{f0}} \quad \frac{a_1\alpha_1}{S\tau_1 + 1} - \frac{a_2\alpha_2}{S\tau_2 + 1} + \frac{a_3\alpha_3}{S\tau_3 + 1} \tag{70}$$

where

$$\tau_1 = \frac{C_1r_1R_1}{r_1 + R_1} \qquad \alpha_1 = \frac{R_f}{r_1 + R_1}$$

$$\tau_2 = \frac{C_2r_2R_2}{r_2 + R_2} \qquad \alpha_2 = \frac{R_f}{r_2 + R_2}$$

$$\tau_3 = \frac{C_3r_3R_3}{r_3 + R_3} \qquad \alpha_3 = \frac{R_f}{r_3 + R_3}$$

it becomes

$$\frac{e_0(s)}{e_{in}(s)} = - \frac{2a_1\alpha_1\tau_1^2}{SR_0C_{f0}} \left[\frac{S^2}{(S\tau_1 + 1)(S\tau_2 + 1)(S\tau_3 + 1)}\right] \tag{71}$$

if the parameters are chosen such that

$$\tau_n = n\tau \tag{72}$$

$$a_1\alpha_1 - a_2\alpha_2 + a_3\alpha_3 = 0 \tag{73}$$

$$5a_1\alpha_1 - 4a_2\alpha_2 + 3a_3\alpha_3 = 0 \tag{74}$$

$$6a_1\alpha_1 - 3a_2\alpha_2 + 2a_3\alpha_3 \neq 0 \tag{75}$$

The solution to the above equations for $A_n\alpha_n$ yields the results

$$a_2\alpha_2 = 2a_1\alpha_1 \tag{76}$$

$$a_3\alpha_3 = a_1\alpha_1 \tag{77}$$

The ratio of the relative gains of the three legs is 1,-2,+1 and these coefficients are those given by Bickley for the three point approximation to a second derivative. The transfer function, Eq. (71), shows the network generates a second derivative, and then integrates this to obtain the first derivative. The analysis of the circuit is identical to that discussed previously for three poles. The noteworth point of this approach is that both the first and second derivative are generated at the same time.

IV. CONCLUSIONS

From mathematical approximations of the type

$$\frac{A^m y_r^{(m)}}{m!} = \frac{1}{P!} \left(A_0 y_0 + A_1 y_1 + \; .... \; A_p y_p \right) + \mathcal{E} \qquad (32)$$

A systematic family of derivative circuits can be derived. These circuits are at least as satisfactory as those normally adopted and in most cases superior for control purposes. Their superiority arises from the fact that they incorporate more poles than is customarily achieved. Thus their "cut-off frequency line" can be made appreciably steeper than the conventional -6db/octave. The slope of this line is $-6(n-1)$db/octave where n = pole number. Associated with this there is an increase of accuracy in derivative value for low-frequency signals.

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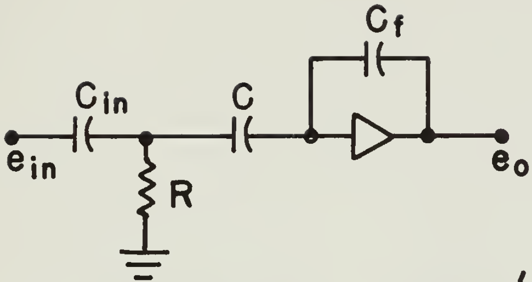
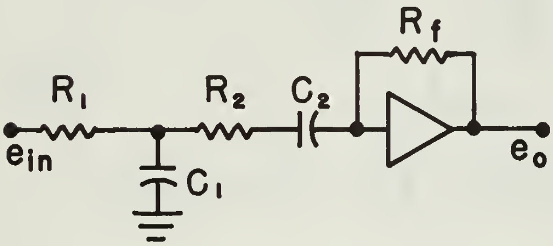
APPENDIX A



$$\frac{e_o\,(S)}{e_{in}\;S} = -\frac{R\,C^2 S}{C_f\,[S(2RC)+1]}$$

(a)



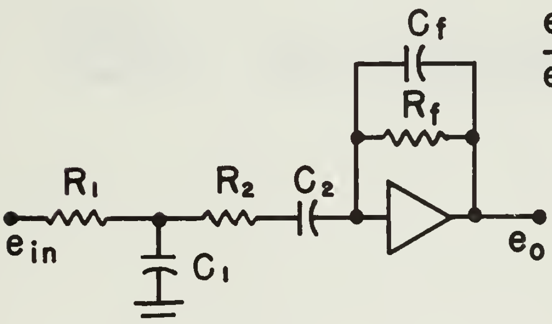
$$\frac{e_o(S)}{e_{in}(S)} = -\frac{R_f C_2 S}{(S\tau_1+1)(S\tau_2+1)}$$

where $\tau_1\;\tau_2 = R_1 R_2 C_1 C_2$

$\tau_1+\tau_2 = R_1 C_1 + R_2 C_2 + R_1 C_2$

$\tau_1 \neq \tau_2$

(b)



$$\frac{e_o(S)}{e_{in}(S)} = -\frac{R_f C_2 S}{(S\tau_1+1)(S\tau_2+1)(S\tau_3+1)}$$

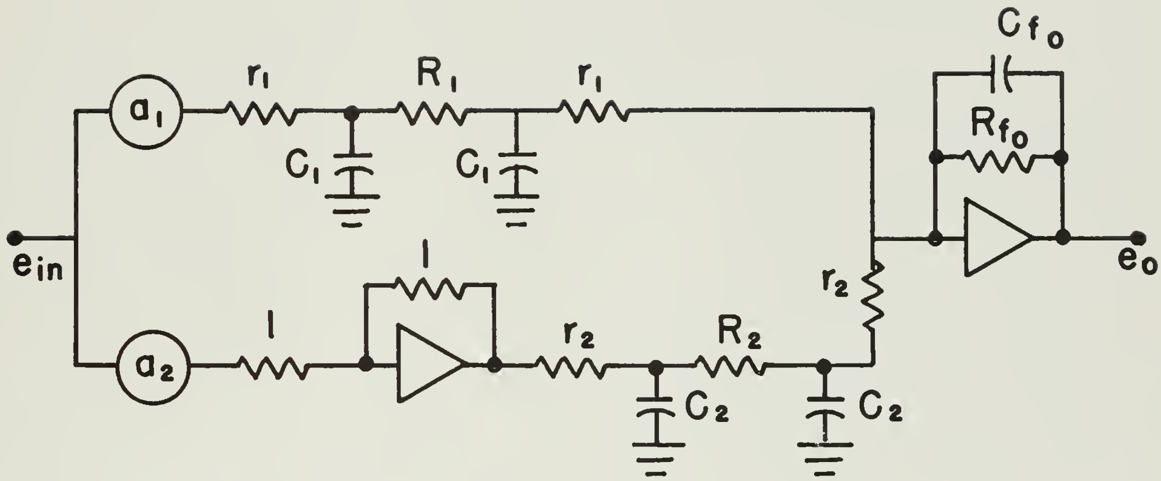
$\tau_1 = R_f\; C_f$

$\tau_2\;\tau_3 = R_1\; R_2 C_1\; C_2$

$\tau_2 + \tau_3 = R_1\, C_2 + R_2\; C_2 + R_1\, C_2$

$\tau_1 \neq \tau_2$

(c)



$$\frac{e_o(S)}{e_{in}(S)} = - \frac{K\,S}{(S\tau_0+1)(S\tau_1+1)(S\tau_2+1)(S\tau_3+1)(S\tau_4+1)}$$

(d)